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VARIATIONS ON FOURIER WAVE THEORY

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SUMMARY

A review of the analytical and numerical background of Fourier wave theory establishes the commonality of existing formulations and identifies a number of analytical and numerical assumptions that are unnecessary. Some formulations in particular lack flexibility in excluding the possibility of Stokes' second definition of phase speed. A generalized formulation is introduced for comparative purposes and it is shown that published solutions differ only in the approach to the limit wave. Detailed consideration of truncation order confirms that it is the crucial parameter, especially at extreme wave heights. All formulations considered are shown to provide acceptable solutions for small to moderately extreme waves.

KEY WORDS Fourier wave theory Limit wave Stream function Truncation order

INTRODUCTION

Developments in computer technology have had a considerable influence on steady progressive wave theory in recent decades. One approach has been the refinement of existing analytical theories to very high orders of approximation through application of symbolic manipulation software or computer algebra. Schwartz[1] and Cokelet[2] in particular have extended the classical Stokes deep water expansion to very high orders (typically over 100) and have been able to extend the theoretical range of validity into reasonably shallow water. Apart from loss of precision in shallow water,[3] this is not an approach that is particularly suitable for routine application in practice.

Direct numerical solutions are an alternative approach, and the boundary integral equation method would appear to be the most promising technique. Unsteady wave problems, especially the approach to breaking, have received considerable attention.[4–6] Steady progressive waves have been considered by Baker *et al.*[7] and Lu *et al.*[8] The problem of imposing the free surface boundary conditions at a boundary which is itself part of the solution is a common difficulty in any formulation, but it appears to be an especially onerous problem for direct numerical methods. As yet also, this approach is not suitable for routine application in practice.

A third approach, however, the Fourier approximation method, seems to offer just the right combination of analytical veracity, numerical robustness and computational efficiency. Required computer resources are consistent with present microcomputer technology, so that this approach is suitable for routine application in practice.

There are however a number of apparently competing formulations of Fourier wave theory[9–11] but no clear guidelines to assist in the selection of an appropriate formulation and in the adoption of a suitable numerical solution algorithm. Fourier wave theory is a hybrid analytical–numerical theory in which the solution is partially analytical in accommodating the field equation and the

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kinematic bottom boundary condition but is completed numerically. Both the analytical and numerical details are important in achieving a complete solution and there is a surprising range of practices in existing formulations. It is the purpose of the present paper to compare and contrast the competing formulations, with particular attention being given to the problem formulation and to solution comparisons. This analysis provides an appropriate framework in which to evaluate the conflicting claims of the alternative formulations.

STEADY WAVE THEORY

Progressive waves of permanent form are steady in a frame of reference moving at the phase speed C. Accordingly, it is convenient to adopt a steady and moving x, z reference frame (Figure 1) that is located at the mean water level (MWL) and moves at speed C with the wave crest, rather than an unsteady and fixed X, Z reference frame. Assuming that the flow is incompressible and irrotational, the mathematical formulation may be presented in terms of the Euler equations, the velocity potential function or the streamfunction. Choosing the streamfunction $\psi(x, z)$, the field equation representing mass and momentum conservation is the Laplace equation

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial z^2}=0, \tag{1}$$

where the velocity components (u, w) are $(\partial\psi/\partial z, -\partial\psi/\partial x)$.

This field equation is subject to the following boundary conditions.

(1) The bottom boundary condition, representing no flow through horizontal bed, is

$$\psi(x, -h)=0 \quad \text{at} \quad z=-h. \tag{2}$$

(2) The kinematic free surface boundary condition (KFSBC), representing no flow through the free surface, is

$$\psi(x, \eta)=-Q \quad \text{at} \quad z=\eta(x), \tag{3}$$

where $\eta(x)$ is the free surface and $-Q$ is the constant volume flow rate per unit width under the steady wave. Q is positive and this flow is in the negative x-direction.

(3) The dynamic free surface boundary condition (DFSBC), representing constant atmospheric pressure on the free surface, is

$$\frac{1}{2}\left(\frac{\partial\psi}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial\psi}{\partial z}\right)^2+g\eta=R \quad \text{at} \quad z=\eta(x), \tag{4}$$

where g is the gravitational acceleration and R the Bernoulli constant.

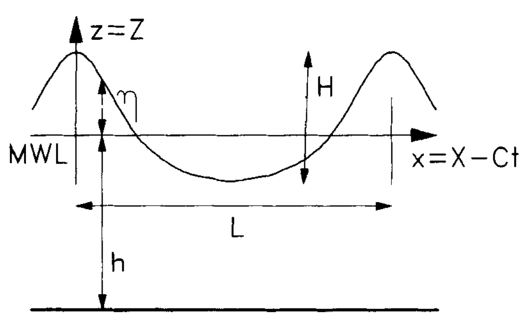

Figure 1. Definition sketch for steady wave theory

(4) The wave is periodic:

$$\psi(x+L, z)=\psi(x, z), \tag{5}$$

where $L(=2\pi/k)$ is the wave length and k is the wave number.

The given parameters defining a steady wave solution are generally the wave height H, the water depth h, the wave period $T(=2\pi/\omega)$ and either the coflowing Eulerian current C_E or the wave-averaged mass transport velocity or Stokes drift C_S. The wave height is defined as

$$H=\eta(0)-\eta(L/2), \tag{6}$$

and mass conservation requires an invariant MWL such that

$$\int_0^{L/2} \eta(x)\mathrm{d}x=0. \tag{7}$$

The speed C of the moving and steady reference frame is related to the fixed and unsteady reference frame by the dispersion relationship. When C_E is known, the dispersion relationship is

$$C=L/T=\bar{u}+C_E, \tag{8}$$

where $-\bar{u}$ is the mean fluid speed at any z wholly within the fluid. The Stokes drift is then defined as

$$C=L/T=Q/h+C_S. \tag{9}$$

When C_S is known, equation (9) is the dispersion relationship and equation (8) is the definition equation for C_E.

FOURIER APPROXIMATION WAVE THEORY

Competing formulations of Fourier wave theory have much in common, and it is initially convenient to introduce what is essentially a superset of these formulations prior to a discussion of the specific variations. The solution for the streamfunction is represented by a truncated Fourier series

$$\psi(x, z)=-\bar{u}(h+z)+\frac{g^2}{\omega^3}\sum_{j=1}^{N} B_j \frac{\sinh jk(h+z)}{\cosh jkh}\cos jkx, \tag{10}$$

where the B_j are the dimensionless Fourier coefficients, of which there are N. This representation of the streamfunction automatically satisfies the field equation, the kinematic bottom boundary condition and the periodic lateral boundary conditions. The Fourier coefficients are chosen numerically to satisfy the free surface boundary conditions, the finite truncation order N being the only necessary assumption of Fourier wave theory.

The unknown variables in a Fourier wave solution are k, $\bar{u}$, C_E or C_S, Q, R, η_m for $m=0\,(1)M$ and B_j, of which there are $M+N+6$. The $\eta_m=\eta(x_m)$ are water surface nodes, where the $x_m=m\pi/kM$ are uniformly distributed in x from crest to trough.

The problem formulation provides $2M+6$ implicit algebraic equations in these $M+N+6$ unknowns, each equation being cast in the form

$$f_i(k, \bar{u}, C_E \text{ or } C_S, Q, R, \eta_m, B_j)=0. \tag{11}$$

The equations define the wave height

$$f_1=\eta_0-\eta_M-H, \tag{12}$$

the mean water level

$$f_2=\frac{1}{2M}\left(\eta_0+2\sum_{m=1}^{M-1}\eta_m+\eta_M\right), \tag{13}$$

the Eulerian current

$$f_3=\frac{2\pi/k}{T}-\bar{u}-C_E, \tag{14}$$

the Stokes drift

$$f_4=\frac{2\pi/k}{T}-\frac{Q}{h}-C_S, \tag{15}$$

the kinematic free surface boundary condition (KFSBC) at each of the $M+1$ free surface nodes

$$f_{5+2m}=\psi(x_m,\eta_m)+Q \tag{16}$$

and the dynamic free surface boundary condition (DFSBC) at each of the free surface nodes

$$f_{6+2m}=\frac{1}{2}\left(\frac{\partial\psi(x_m,\eta_m)}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial\psi(x_m,\eta_m)}{\partial z}\right)^2+g\eta_m-R. \tag{17}$$

Note in particular the use of the trapezoidal rule in equation (13) for the MWL. This is an exact result for the continuous integral in equation (7), where $\eta(x)$ is represented by a truncated Fourier series, as is implied by equation (10).

The problem is uniquely defined for $M=N$ and overspecified for $M>N$. The solution of a set of $2N+6$ simultaneous implicit algebraic equations in $2N+6$ unknowns is a familiar problem in numerical analysis for which successful algorithms are generally variations on the Newton–Raphson method. A set of $2M+6$ simultaneous implicit algebraic equations in $M+N+6$ unknowns, where $M>N$, is an equally familiar problem in numerical analysis in the context of non-linear optimization. A solution is established by seeking a minimum value for an objective function of the $M+N+6$ unknowns. A familiar algorithm is the least-squares method where the objective function is the sum of squares of the left-hand sides of the $2M+6$ equations:

$$O(k,\bar{u},C_E \text{ or } C_S,Q,R,\eta_m,B_j)=f_1^2+f_2^2+\ldots+f_{2M+6}^2. \tag{18}$$

Such an algorithm is equally successful for $M=N$, where the objective function would be expected to be zero. In practice, this involves little sacrifice in computational efficiency and none in solution precision and is accordingly a convenient choice of algorithm for the present purposes.

The choice of numerical solution algorithm should not influence the solution, and the present computations have exclusively adopted the IMSL subroutine ZXSSQ which is a finite difference Levenberg–Marquardt algorithm with strict descent in double precision. This algorithm is mature, routinely successful and commonly available. Given that a solution exists, there are two potential difficulties with any optimization algorithm. The first is the difference in physical dimensions and relative magnitudes of the dependent variables. This has been minimized by redefining the variables and the implicit algebraic equations in dimensionless form, here in terms of ω and g. A second difficulty is the potential existence of multiple solutions, especially the odd harmonics which are legitimate mathematical solutions of the gravity wave problem as formulated. This problem can be avoided, for example, by the choice of an initial estimate of the complete solution from Airy wave theory at a fraction of the given wave height. The wave height is then progressively increased towards the given wave height, with an initial estimate at each subsequent step being provided by a Taylor series expansion in H about the converged solution at the previous height step. A fraction sequence of 0·01, 0·02, 0·05, 0·1, 0·2, 0·5, 1·0 has been employed, with two steps normally sufficient in very deep water, four in transitional water and all seven in extremely shallow water.

Whether a solution exists at all is a further potential difficulty, and there are two aspects here that require attention. The first is the truncation order N of the Fourier series. Steep crest and flat trough profiles typical of shallow water waves require many more Fourier terms than the more closely sinusoidal wave profiles in deep water. The theoretical slope discontinuity at the crest of limit waves would require an infinite truncation order and cannot be accommodated by Fourier wave theory; in practice, however, adequate solutions can be achieved very close to this limit. The second aspect is whether or not a solution does indeed exist, and here the problem formulation is remarkably prophetic and robust. For combinations of the given parameters that are not physically possible, convergence was consistently not achieved with the present formulation, despite the mathematical possibility of a minimum of the objective function. This is an especially encouraging aspect of the problem formulation and the numerical solution, considering the extreme multidimensionality of the problem and the considerable potential for spurious solutions.

The forms of the free surface boundary conditions utilized above are the primitive forms, but equally appropriate forms may include the MWL constraint together with the truncated Fourier summation. The KFSBC, for example, may be written

$$-\bar{u}(h+\eta(x))+K(x)=-Q \quad \text{for all } x, \tag{19}$$

where

$$K(x)=\frac{g^2}{\omega^3}\sum_j B_j \frac{\sinh jk(h+\eta)}{\cosh jkh}\cos jkx.$$

Averaging by integration in x over a symmetric half-wave gives

$$-\bar{u}h+\frac{2}{L}\int_0^{L/2} K(x)\mathrm{d}x=-Q, \tag{20}$$

where the integral of the initial term is zero from the MWL constraint. This is now a global or weak form of the KFSBC since it enforces the boundary condition at all x only in an averaged sense. Equation (19), and equations (3) and (16), are strong forms of the boundary condition. Note however that Q may be eliminated between equations (19) and (20), giving

$$-\bar{u}\eta(x)+K(x)=\frac{2}{L}\int_0^{L/2} K(x)\mathrm{d}x, \tag{21}$$

where $\bar{K}$ may be written for the right-hand side. This remains a strong form of the KFSBC.

Similarly, the DFSBC may be written

$$D(x)+g\eta(x)=R \quad \text{for all } x, \tag{22}$$

where

$$D(x)=\frac{1}{2}\left(\frac{g^2}{\omega^3}\sum_j jkB_j\frac{\sinh jk(h+\eta)}{\cosh jkh}\sin jkx\right)^2+\frac{1}{2}\left(\bar{u}+\frac{g^2}{\omega^3}\sum_j jkB_j\frac{\cosh jk(h+\eta)}{\cosh jkh}\cos jkx\right)^2,$$

which is a strong form. Integration over x gives the weak form

$$\frac{2}{L}\int_0^{L/2} D(x)\mathrm{d}x=R, \tag{23}$$

where the MWL constraint has again been used. Eliminating R between these two equations gives the alternative strong form

$$D(x)+g\eta(x)=\frac{2}{L}\int_0^{L/2} D(x)\mathrm{d}x, \tag{24}$$

where $\bar{D}$ may be written for the right-hand side.

DEAN FORMULATION

The Dean formulation[9, 12] imposes the Stokes first definition of phase speed and has a simpler definition of the Fourier sum, representing the streamfunction as

$$\psi(x,z)=\left(\frac{2\pi/k}{T}-C_{\mathrm{E}}\right)(h+z)+\sum_{j=1}^{N} A_j \sinh jk(h+z)\cos jkx. \tag{25}$$

Flexibility is lost in excluding the Stokes second definition of phase speed which would be necessary for wave computations in the near-shore zone and in wave flumes. The Fourier coefficients are

$$A_j=\frac{g^2 B_j}{\omega^3 \cosh jkh}, \tag{26}$$

which is dimensional and includes the hyperbolic cosine factor, both of which may lead to numerical precision problems.

Unlike the generalized formulation outlined previously, the Dean formulation does not seek a simultaneous solution for all dependent variables. Instead, it adopts a four-step procedure which progressively solves for k and A_j at step I, η_m at step II, Q and C_{S} at step III and finally R at step IV. At each step the remaining dependent variables are held constant at their most recent value. As for the generalized algorithm, an initial estimate of the solution is required, which is identified by the superscript zero.

At step I, the unknowns are k^{I} and the A_j^{I}, which number $N+1$ and are identified by the superscript I. The solution is based on the DFSBC, the f_{6+2m} equations, which can be written

$$f_{6+2m}=D(x_m, k^{\mathrm{I}}, A_j^{\mathrm{I}}; \eta_m^0)+g\eta_m^0-R^0. \tag{27}$$

The adopted objective function is

$$O(k^{\mathrm{I}}, A_j^{\mathrm{I}})=f_6^2+f_8^2+\ \dots\ +f_{6+2M}^2. \tag{28}$$

As before, a least-squares solution algorithm is appropriate for $M>N$. In the Dean tables,[12] N ranges from 2 in deep water to 19 in shallow water. M is not explicitly given but appears to be 36 (or $36n$, where n is a positive integer) throughout.

At step II, the unknowns are the η_m^{II}, which number $M+1$ and are identified by the superscript II. The solution is based on the KFSBC, the f_{5+2m} equations, which can be written

$$f_{5+2m}=\psi(x_m, \eta_m^{\mathrm{II}}; k^{\mathrm{I}}, A_j^{\mathrm{I}})+Q^0. \tag{29}$$

In fact, η_m^{II} is the sole unknown in each of these $M+1$ equations, although the equations remain implicit. These equations may be solved separately by familiar algorithms such as interval halving, *regula falsi* or Newton–Raphson.

At step III, Q^{III} is estimated from the weak form of the KFSBC, equation (20), which incorporates the MWL constraint, the f_2 equation, as

$$Q^{\mathrm{III}}=-\bar{K}(k^{\mathrm{I}}, \eta_M^{\mathrm{II}}, A_j^{\mathrm{I}}), \tag{30}$$

where the $\bar{K}$ integral is determined by Simpson's rule. C_S^{III} is available explicitly from equation (9) (or the f_4 equation) as

$$C_S^{III} = \frac{Q^{III}}{h} - \frac{2\pi/k^I}{T}. \tag{31}$$

At this stage, wave height has not been imposed on the solution since Dean chose to impose R rather than H. The f_1 equation is introduced at step IV and may be written

$$f_1 = \eta_0^{II}(R^0) - \eta_m^{II}(R^0) - H. \tag{32}$$

This is a single implicit algebraic equation in R which can be solved in an iterative manner as in step II. An iteration cycle involves repetition of steps I, II and III.

In the generalized formulation, the single assumption was the truncation order N. Dean has introduced two further assumptions. The first is Simpson's rule integration in step III, whereas trapezoidal rule integration is exact in the context of truncated Fourier series. The second is the multistep algorithm, which does not provide a simultaneous solution for all dependent variables. While both of these assumptions are unnecessary, it does not appear that they have a detrimental influence on the algorithm.

Perhaps the major difficulty with the multistep algorithm is identified at step IV, which implicitly assumes that H is a single-valued function of R. For small to moderately high waves this is the case, but H is in fact a double-valued function of R for waves of near-limiting height.[1,2] All of the common integral properties of steady waves are in fact dual-valued near the limiting wave height. Chaplin[10] has shown that the Dean tabulated solutions[12] do not follow this trend and concludes accordingly that the Dean limiting wave height solutions (case D) have little credibility. The tabulated Dean solutions for cases A, B and C for small to moderately steep waves are not disputed.

It is emphasized in the Dean presentation[9,12] that the KFSBC is exactly satisfied, but this is a misleading statement. The Fourier series representation for the streamfunction does exactly satisfy the field equation and the kinematic bottom boundary condition. In step II of the Dean algorithm, η_m^{II} is indeed chosen as the numerical solution of the implicit algebraic equation (29), but it explicitly assumes that the balance of the problem formulation can be decoupled. In particular, R, Q and the A_j are assumed to be given parameters, equated to Q^0 from the initial solution estimate and k^I and A_j^I from step I. This is a substantially weaker statement that cannot be categorized as exact; it is a numerical approximation. Numerical errors will remain in the KFSBC, albeit small.

Dalrymple[13] introduced a variation on the Dean formulation, also adopted by Huang and Hudspeth,[14] that combines steps I and IV and avoids the final iteration on R to impose the wave height. Dalrymple adopts the alternative strong form of the DFSBC, equation (24), which can be written

$$f_{6+2m} = D(x_m, k^I, A_j^I; \eta_m^0) + g\eta_m^0 - \bar{D}(k^I, A_j^I; \eta_m^0), \tag{33}$$

where the $\bar{D}$ integral is determined by Simpson's rule and the objective function at step I is redefined as

$$\begin{aligned} O(k^I, A_j^I, \lambda_1, \lambda_2) &= \varepsilon_{DFSBC}^2 + \varepsilon_H + \varepsilon_{MWL} \\ &= \frac{1}{M+1}\sum_{m=0}^{M} f_{6+2m}^2 + \lambda_1(\eta_0^0 - \eta_M^0 - H) + \lambda_2\frac{1}{3M}\sum_{m=0,2,4,\ldots}^{M-2}(\eta_m + 2\eta_{m+1} + \eta_{m+2}), \end{aligned} \tag{34}$$

which includes the wave height constraint (the f_1 equation) as the second term and the MWL constraint (the f_2 equation) evaluated by Simpson's rule as the final term; λ_1 and λ_2 are unknown multipliers. Steps II and III remain unchanged, except for the addition to step III of the estimation of the Bernoulli constant from the weak form of the DFSBC, equation (23), as

$$R^{\mathrm{III}} = \bar{D}(k^{\mathrm{I}}, \eta_m^{\mathrm{II}}, A_j^{\mathrm{I}}), \tag{35}$$

where Simpson's rule is again used for the numerical integration.

There are a number of inconsistencies in this modification to the Dean formulation. The MWL constraint is included twice in the revised objective function; it appears implicitly in the equation (33) version of the DFSBC as well as explicitly as the last term in equation (34). In addition, Simpson's rule is used in the equation (33), (34) and (35) integrations as well as in step III for Q.

The major uncertainty perhaps is the definition of the objective function itself. Normalized variables are apparently not used and the additional f_1 and f_2 equations are not squared, so that they are weighted differently to the DFSBC equations. Further, in seeking a minimum of the objective function, the DFSBC equations can contribute only positive residuals, whereas the f_1 and f_2 equations can contribute positive or negative residuals. Residuals of different sign to the associated λ multipliers can potentially result in a spurious minimum of the objective function and a spurious steady wave solution. The λ multipliers are additional dependent variables and there can be no expectation that a multidimensional iterative algorithm such as Newton–Raphson will yield solutions for λ_1 and f_1 and also λ_2 and f_2 that will always have the same sign.

Huang and Hudspeth[14] acknowledge the weighting and sign problems in adopting a convergence criterion based on a dimensional total error

$$\varepsilon_{\mathrm{T}} = \varepsilon_{\mathrm{DFSBC}} + |\varepsilon_H| + |\varepsilon_{\mathrm{MWL}}|, \tag{36}$$

which differs from the equation (34) objective function on which iteration continues to be based. From a numerical viewpoint, it would appear that the Dalrymple and the Huang and Hudspeth variations on the Dean formulation introduce a number of additional uncertainties that may lead to spurious solutions. These uncertainties must be balanced against the numerical convenience of avoiding the step IV iteration for the wave height in the Dean formulation. On balance, the original Dean formulation must remain preferable for small to moderately extreme wave heights, despite the additional computational effort.

CHAPLIN FORMULATION

Chaplin[10] also adopts the simpler equation (25) form of the truncated Fourier series for the streamfunction, but without the C_{E} term. Flexibility is again lost in excluding the Stokes second definition of phase speed and, further, in excluding a coflowing current (the C_{E} term) from the Stokes first definition of phase speed. The 1/cosh jkh factor is also omitted. As with the Dean formulation, Chaplin does not seek a simultaneous solution for all dependent variables but adopts a three-step procedure that solves for η_{M-1} and η_M at step I, Q and the A_j at step II and finally k and $\eta_0, \eta_1, \eta_2, \ldots, \eta_{M-2}$ at step III. C_{S} and R are computed upon completion of step III.

At step I, the unknowns are η_{M-1}^{I} and η_M^{I} which are estimated from the wave height and MWL constraints. The wave height constraint (the f_1 equation) gives

$$\eta_M^{\mathrm{I}} = \eta_0^0 - H. \tag{37}$$

Simpson's rule integration is used in the MWL constraint (the f_2 equation) which, together with the wave height constraint, gives

$$\eta_{M-1}^{\mathrm{I}} = \frac{1}{2}\left(\eta_{M-2}^0 + \eta_0^0 - H + \sum_{m=0,2,4,\ldots}^{M-4} (\eta_m^0 + 2\eta_{m+1}^0 + \eta_{m+2}^0)\right). \tag{38}$$

At step II, the unknowns are Q^{II} and the A_j^{II}, which number $N+1$. The solution is based on the KFSBC, the f_{5+2m} equation, which can be written

$$f_{5+2m} = -\frac{2\pi/k^0}{T} + K(x_m, A_j^{\text{II}}; k^0, \eta_0^0, \eta_1^0, \ldots, \eta_{M-2}^0, \eta_{M-1}^1, \eta_M^1) - Q^{\text{II}}. \tag{39}$$

There are $M+1$ such equations. Chaplin chooses M to significantly exceed N and solves equation (39) by generalized Fourier analysis, a set of orthonormal functions being established by the Schmidt process. A least-squares algorithm would be equally appropriate. Chaplin chooses M as 200; N values range from 9 to 51.

At step III, the unknowns are k^{III} and $\eta_0^{\text{III}}, \eta_1^{\text{III}}, \ldots, \eta_{M-2}^{\text{III}}$, which number M. The solution is based on the DFSBC (the f_{6+2m} equations) which can be written

$$f_{6+2m} = D(x_m, k^{\text{III}}, \eta_m^{\text{III}}; A_j^{\text{II}}) + g\eta_m^{\text{III}} - \bar{D}(k^{\text{III}}, \eta_0^{\text{III}}, \eta_1^{\text{III}}, \ldots, \eta_{M-2}^{\text{III}}; \eta_{M-1}^1, \eta_M^1, A_j^{\text{II}}). \tag{40}$$

There are $M+1$ such equations. The problem is again overspecified and Chaplin chooses a least-squares algorithm. Upon completion of this numerical solution, C_S is available from the f_4 equation as

$$C_S^{\text{III}} = \frac{Q^{\text{III}}}{h} - \frac{2\pi/k^{\text{III}}}{T}, \tag{41}$$

and R from the weak form of the DFSBC, Equation (23), as

$$R^{\text{III}} = \bar{D}(\eta_{M-1}^1, \eta_M^1, A_j^{\text{II}}, k^{\text{III}}, \eta_0^{\text{III}}, \eta_1^{\text{III}}, \ldots, \eta_{M-2}^{\text{III}}). \tag{42}$$

This multistep sequence is an approximate inversion of the Dean formulation. Dean solves for the A_j from the DFSBC and then the η_m from the KFSBC, whereas Chaplin solves for the A_j from the KFSBC and then the balance of the η_m from the DFSBC; in both cases, k is determined from the DFSBC. The Chaplin sequence does involve additional computational effort since both the KFSBC and DFSBC steps are now simultaneous multidimensional problems, whereas the Dean KFSBC step is a sequence of parallel one-dimensional problems. Computational effort, however, is not a major concern since these solutions can be completed on a microcomputer. Otherwise the Chaplin formulation would appear to be preferable in its accommodation of wave height, especially since it is apparently capable of extension to near-limit wave heights.

Several of the criticisms of the Dean formulation persist, namely the adoption of Simpson's rule integration for the MWL constraint and the fact that the multistep sequence does not provide a simultaneous solution for all dependent variables. Again, these assumptions are unnecessary but do not appear to have a detrimental influence on the algorithm.

FENTON FORMULATION

The Fenton formulation[11,15] forms the basis of the generalized formulation outlined above. There are two minor differences. Fenton chooses to normalize the equations in terms of k and g rather than ω and g. The present choice of ω and g simplifies the coding of the problem formulation since k is a dependent variable, but is otherwise equivalent. In addition, M is chosen equal to N so that the problem is exactly specified and a Newton–Raphson-style algorithm can be used. Huang and Hudspeth[14] dismiss the Fenton formulation on this point, citing 'large discrepancies between fifth-order results compared by Rienecker and Fenton[11] as only five points on the wave profile were used for convergence', but this is not an accurate reflection of the manuscript. The paper in fact reports solutions at truncation orders of 8, 16 and 32 for conditions defined by experiments of Le

Mehaute *et al.*[16] These solutions are apparently almost identical and there are no solutions at truncation orders less than 8.

The adoption of an adequate truncation order N is in fact a potential criticism of all formulations of Fourier wave theory, including that recommended by Huang and Hudspeth, and not just the Fenton formulation. In the generalized formulation, the only assumptions are the choice of N and M, where $M \geqslant N$. The success of a steady wave solution is then clearly dependent on N and M.[17] Dean,[9] Dalrymple[13] and Huang and Hudspeth[14] do not record M values, beyond a statement to the effect that M is large. The Dean tables[12] have $M = 36n$, where n is an unrecorded positive integer. Chaplin[10] apparently uses $M = 200$. Rienecker and Fenton[11] explicitly adopt $M = N$ and demonstrate increasing precision for $N = 8$, 16 and 32, the results being indistinguishable from the Cokelet tables[2] at $N = 32$ in shallow water and at $N = 8$ in deep water. That N rather than M is the crucial parameter has been confirmed in a rather more detailed analysis by Sobey,[17] which has shown that the Fenton choice of $M = N$ is appropriate provided that N is adequate for the particular steady wave solution.

The significance of the truncation order follows directly from the nature of Fourier series approximations to periodic profiles. High waves in deep water have sharp crest profiles and flat trough profiles. High fidelity requires a larger truncation order, for which additional surface nodes ($M > N$) will not compensate. Limiting wave heights have a theoretical slope discontinuity at the crest, which demands an infinite truncation order. Such is not possible and no Fourier wave formulation can represent more than near-limit wave conditions, as is clearly acknowledged by Chaplin[10] and Rienecker and Fenton.[11]

COMPARISONS

Fourier wave theory is a hybrid analytical–numerical theory and the sufficiency of both the analytical and numerical phases are relevant concerns. Existing formulations have rather more in common than is perhaps apparent from published descriptions. This is illustrated by a comparison of tabulated Dean solutions[9] for case 3C ($\omega^2 h/g = 2\pi/100$; $\omega^2 H/g = 0{\cdot}036\,57$, $C_E = 0$; $N = 17$, $M = 36$) and case 7C ($\omega^2 h/g = 2\pi/5$; $\omega^2 H/g = 0{\cdot}589\,27$, $C_E = 0$; $N = 7$, $M = 36$) with the present generalized formulation, in Figures 2 and 3 respectively. The continuous lines are the

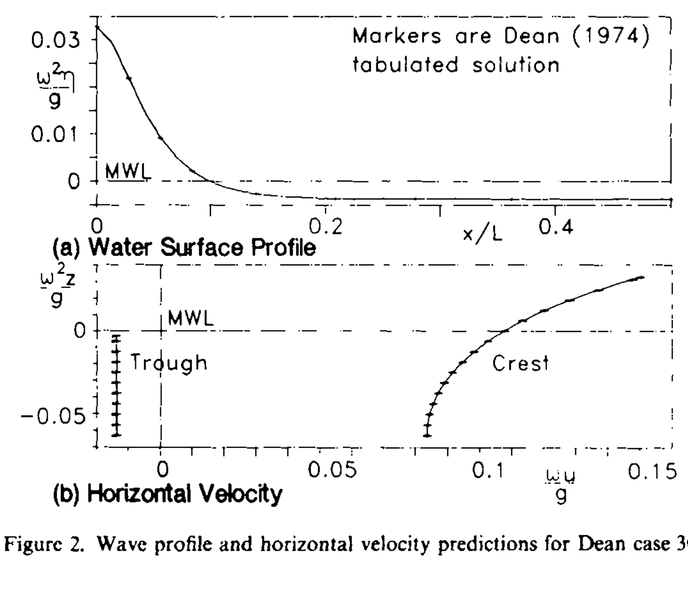

Figure 2. Wave profile and horizontal velocity predictions for Dean case 3C

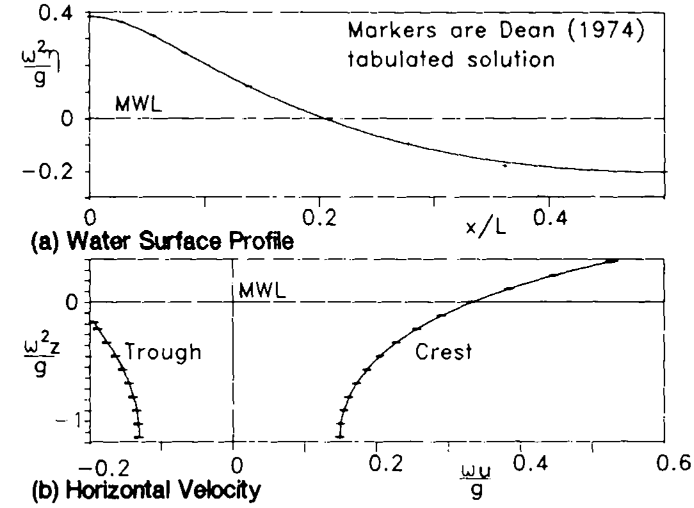

Figure 3 Wave profile and horizontal velocity predictions for Dean case 7C

present solutions and the markers are the tabulated Dean solutions, parts (a) being the water surface profile and parts (b) the horizontal velocity profiles under the crest and the trough. The solutions are visually identical, with the single exception of the WS elevation listed in the Dean tables for case 7C at $x/L = 130°$. The otherwise visually perfect agreement would indicate that the Dean table entry was a misprint. The table entry would appear to be appropriate for $x/L = 140°$. This in fact may be a consistent misprint throughout the Dean tables, which should be apparent in all deep water solutions but not in shallow water solutions such as Figure 2(a) where the trough profile is long and flat.

It follows from these and other similar comparisons for case A, B and C waves (respectively 25%, 50% and 75% of an empirical breaking wave height) that the numerical approximations adopted by Dean do not compromise the utility of his solutions for small to moderate wave heights. Lack of flexibility, however, remains a concern, the Dean tables assuming the Stokes first definition of phase speed with $C_E = 0$ and including neither finite C_E nor the Stokes second definition of phase speed.

Insufficient detail of alternative solutions has been published to enable completely satisfactory comparisons. Dalrymple[13] includes little solution detail but does not indicate any disagreement with the Dean solutions. Chaplin[10] and Huang and Hudspeth[14] compare selected integral parameters with the Dean tables, especially Dean cases 3 ($\omega^2 h/g = 2\pi/100$) and 7 ($\omega^2 h/g = 2\pi/5$). As previously commented, the Chaplin solutions confirm the Dean solutions for cases, A, B and C but not for near-limit waves. Similarly, the selected integral parameters tabulated by Huang and Hudspeth differ from the Dean tables only for near-limit waves.

Difficulties are encountered by all formulations as the limiting wave height is approached. This must be expected from a purely physical viewpoint and cannot be avoided. Near-limiting wave heights can be accommodated, but the sharp crest and flat trough profiles require a high truncation order. The slope discontinuity at the crest of limit waves requires an infinite truncation order, and a finite truncation order will encounter the Gibbs phenomenon and solution oscillations. These oscillations can be numerically damped to achieve numerical convergence, but this potentially compromises the physics of the problem formulation, especially where N is small.

For wave heights above the limit waves, the problem formulation is sufficiently robust that no solution can be achieved; at least this is the experience with the generalized formulation.

Differences between simultaneous and multistep numerical algorithms might be expected for near-limit waves because of the rapidly changing nature of the solution. In principle, simultaneous solutions should cope better here, but even this is compromised by the higher order (approximately $2M$) of simultaneous algorithms compared with multistep algorithms (approximately M for the separate KFSBC and DFSBC steps). There is little potential value, however, in pursuing this question since it is clear that Fourier wave theory, regardless of the formulation, is only moderately appropriate for near-limit waves.

This is well illustrated by the approach to the limiting wave height for Dean cases 3 and 7 (where $C_E = 0$), which have been specifically considered by Chaplin and by Huang and Hudspeth. Figure 4 includes normalized phase speed ($\omega C/g$) solutions for Dean case 3 by Dean,[12] by Chaplin[10] and by Huang and Hudspeth.[14] Also shown in Figure 4 are computed solutions from the present generalized algorithm at $N = M = 27$. The point clusters marked A, B and C identify Dean cases 3A, 3B and 3C respectively and further confirm the commonality of all formulations for small to moderately large wave heights. The solutions differ, however, in the approach to the limit wave height. The Dean solution for case 4D is very different to the other solutions, as previously noted by Chaplin. The inset magnifies the detail at the approach to the limit wave and adds further solutions from the present generalized algorithm for $N = M = 18$. The variation is considerable and it is clear that these differences are attributable directly to the numerical algorithm and not to the physical problem. Tabulated solutions by Cokelet[2] provide useful qualitative comparisons but are not sufficiently close to the same $\omega^2 h/g$ conditions to provide useful quantitative comparisons. They do however illustrate the well-known maximum before the limit wave height, which is an intrinsic property of most integral wave parameters. This maximum is reproduced by the Chaplin solution at $N = 49$, by the Huang and Hudspeth solution at $N = 27$ and by the present algorithm at $N = 27$ but not at $N = 18$. It would appear, however, that it is not captured by the Dean algorithm. The direct dependence on truncation order was anticipated and is quite explicit in the inset. Both the location and the magnitude of the peak are dependent on the truncation order. The differences are not large, but high fidelity demands a high truncation order for near-limit waves. Rienecker and Fenton[11] report direct comparisons with the Cokelet tables and also note the significance of the truncation order.

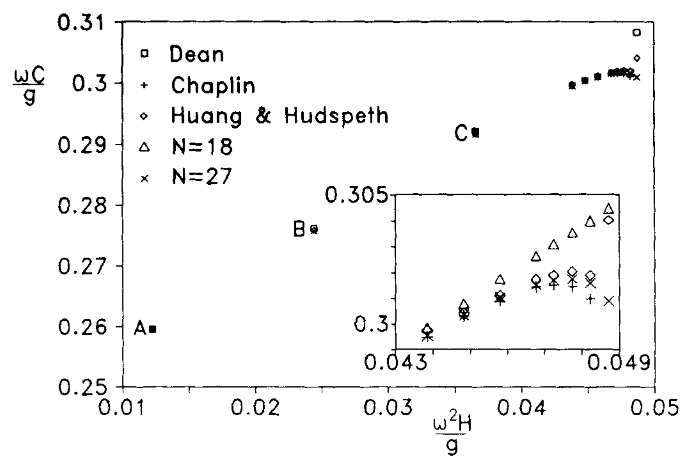

Figure 4. Phase speed dependence on wave height for Dean case 3

Different measures of solution applicability appear to give conflicting guidance for such near-limit waves. On the basis of KFSBC errors, Huang and Hudspeth argue that 'both the Chaplin and Dean solutions are less reliable in steep water wave regions' than their own solutions. In the case of the Dean solutions, this is indeed supported by Figure 4, but the same figure indicates a clear preference for the Chaplin solution. Further, comparison of the multistep Huang and Hudspeth solution at $N=27$ with the present simultaneous solution also at $N=27$ indicates that the multistep algorithm copes less well for near-limit waves.

Similar trends are apparent in Figure 5, which shows Dean solutions, Chaplin solutions at $N=39$, Huang and Hudspeth solutions at $N=13$ and solutions from the present algorithm at $N=18$. The Huang and Hudspeth solutions at $N=13$ do not capture the solution maximum, the truncation order apparently being too low. The present algorithm at $N=18$ and the Chaplin solutions at $N=39$ are very close. The behaviour of the present algorithm on close approach to the limit wave is quite instructive. There is a second curvature reversal and the curve goes high, above the apparent trend of the Dean solutions. A similar trend is observed in the Huang and Hudspeth solutions in Figure 4. These would appear to be spurious solutions, contaminated by the Gibbs phenomenon in an attempt to represent a very steep profile with insufficient Fourier terms. For still higher waves, convergence was not achieved. This spurious behaviour is well illustrated in the response of the Fourier coefficients to increasing wave height, as seen in Figure 6. The Fourier

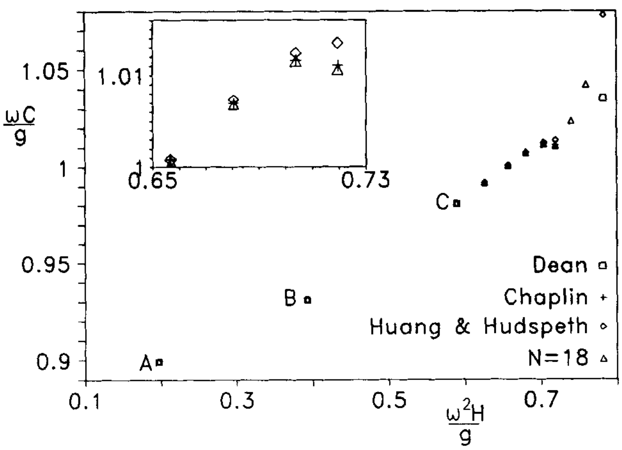

Figure 5. Phase speed dependence on wave height for Dean case 7

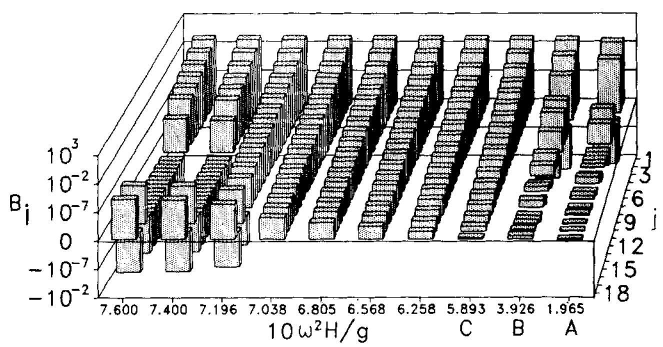

Figure 6. Fourier coefficient dependence on wave height for Dean case 7

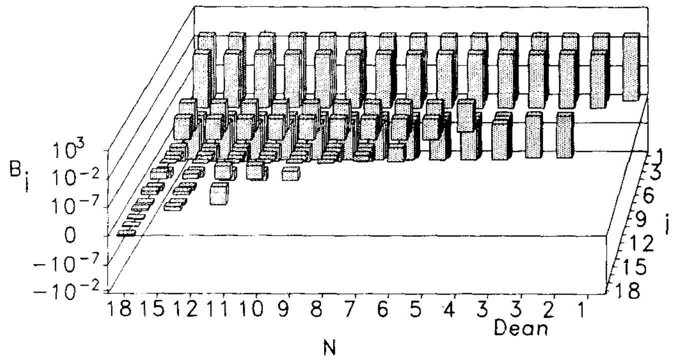

Figure 7. Fourier coefficient dependence on truncation order for Dean case 7A

coefficients for Dean cases 7A, 7B and 7C differ little from those of Dean.[12] The increasing steepness of the crest profile requires increasing contributions from the higher harmonics, leading inevitably to Gibbs phenomenon oscillations of the very high harmonics and an apparent breakdown in the solution fidelity, represented by the eight consecutive negative Fourier coefficients.

For smaller waves, there is perhaps some indication from the sign reversals in the case 7A and 7B Fourier coefficients in Figure 6 that the truncation order for these solutions may be too large. This may conceivably arise when the higher-order dimensionless Fourier coefficients are so small that they exceed (underflow) the finite limits of machine resolution. This was further investigated by computing solutions from the present algorithm at increasing truncation orders from $N=1$ to $N=18$, where M was always chosen equal to N. The computed Fourier coefficients are presented in Figure 7, together with the equivalent Dean[12] coefficients for $N=3$ and $M=36n$. The individual Fourier coefficients for $j \leq N$ are very stable and agree closely with the Dean solution at the same truncation order. It follows that the sign reversals here are not a spurious computational effect. This behaviour, however, is certainly the exception rather than the rule, the general trend being for positive and monotonically decreasing coefficients. Case 7B has similar sign reversal to cases 6B and 7B in the Dean tables. The Fourier coefficients for the $N=M=27$ solutions for Dean case 3 from the present algorithm remain well behaved, monotonically decreasing and positive right up to case 3D. The Figure 4 solution remained similarly well behaved.

CONCLUSIONS

Alternative formulations of Fourier wave theory by Dean,[12] by Dalrymple[13] and Huang and Hudspeth,[14] by Chaplin,[10] by Rienecker and Fenton[11] and Fenton,[15] together with a generalized formulation introduced in the present paper, provide almost identical solutions for small to moderate wave heights over the entire practical range of water depths. The Dean, Dalrymple, Chaplin, and Huang and Hudspeth formulations adopt some numerical approximations such as Simpson's rule integration and a multistep solution algorithm that are not entirely necessary but do not seem to compromise the adequacy of their solutions for non-extreme wave heights. These same formulations, however, lack flexibility in not accommodating the Stokes second definition of phase speed. Both definitions of phase speed together with non-zero values of the mean Eulerian current or the Stokes drift are included in the Rienecker and Fenton formulation and the present generalization of that formulation.

Computed solutions are in rather less agreement at the approach to the limit wave. In principle, and in practice, no Fourier wave theory (with a finite truncation order) can represent the theoretical slope discontinuity at the crest of a limit wave. It is also difficult but not impossible to capture the maximum value of common integral parameters with increasing wave height. The maximum is located at close approach to the limit wave, but this solution characteristic can be filtered from the numerical solution if the adopted truncation order is not sufficiently high. The Dean algorithm is dual-valued for near-limit waves, and the tabulated Dean case D solutions are potentially spurious. The other algorithms appear to have the capability of predicting this feature, provided the truncation order is sufficiently high, although there is some indication that multistep algorithms have somewhat more difficulty.

It is clear from the generalized formulation that the only necessary assumptions in Fourier wave theory are the truncation order N and the number of surface nodes M, which must equal or exceed N. It is not surprising then that the truncation order is the crucial parameter in establishing a successful solution. Of those alternative formulations considered, most have the potential to yield acceptable mathematical solutions for steady non-linear progressive waves, provided the truncation order is appropriate. The exception is the Dean formulation for very high waves.

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